Θέμα 3° / ΕΜΕ 2018

$f(x) = \ln x - \frac{e}{x} + x, \quad x > 0$

α) $f'(x) = \frac{1}{x} + \frac{e}{x^2} + 1 > 0 \Rightarrow f \uparrow, (0,+\infty)$

$f''(x) = -\frac{1}{x^2} - \frac{2e}{x^3} < 0 \rightarrow f \frown, (0,+\infty)$

$\lim_{x\to 0^+} \ln x = -\infty, \quad \lim_{x \to 0^+}\left(-\frac{e}{x}\right) = -\infty \Rightarrow$

$\lim_{x\to 0^+} f(x) = -\infty \Rightarrow$ κατακόρυφη ασύμπτωτη η ευθεία $x=0$ (άξονας $y'y$)

$\lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty}\left(\frac{\ln x}{x} - \frac{e}{x^2} + 1\right) = 1$

$\lim_{x\to+\infty} \frac{\ln x}{x} = \lim_{x\to+\infty} \frac{\frac{1}{x}}{1} = 0, \quad \lim_{x\to+\infty}\left(-\frac{e}{x^2}\right) = 0$

$\lim_{x\to+\infty}(f(x) - x) = \lim_{x\to+\infty}\left(\ln x - \frac{e}{x}\right) = +\infty$

όχι ασύμπτωτες στο $+\infty$.

β)(i) $f\uparrow \Rightarrow f$ 1-1 $\Rightarrow$ αντιστρέψιμη

$D_{f^{-1}} = f(A) = \left(\lim_{x\to0^+} f(x), \lim_{x\to+\infty} f(x)\right) = (-\infty, +\infty) = \mathbb{R}$

(ii) $f^{-1}(2e - f(x)) > e$. Πρέπει $x \in D_f$ και $2e - f(x) \in D_{f^{-1}}$

$\boxed{x>0}$

$f(f^{-1}(2e - f(x))) > f(e) \Leftrightarrow$

$2e - f(x) > e \Leftrightarrow$

$-f(x) > -e \Leftrightarrow$

$f(x) < e \Leftrightarrow f(x) < f(e) \overset{f\uparrow}{\Leftrightarrow}$

$x < e$. Άρα

$x \in (0, e)$

(γ) $\ln x - (ex+1)\cdot\left(\frac{e}{x} - \frac{1}{e}\right) = 2, \quad \boxed{x>0}$

$\ln x - e^2 + x - \frac{e}{x} + \frac{1}{e} = 2 \Leftrightarrow$

$\ln x - \frac{e}{x} + x = e^2 - \frac{1}{e} + 2 \Leftrightarrow f(x) = f(e^2) \Leftrightarrow x = e^2$

(δ) $g(x) = f(x) + f\left(\frac{1}{x}\right), \quad x > 0$

(ii) $g(x) = \cancel{\ln x} - \frac{e}{x} + x - \cancel{\ln x} - ex + \frac{1}{x} =$

$= -e\left(\frac{1}{x} + x\right) + \left(x + \frac{1}{x}\right) = \left(x + \frac{1}{x}\right)(1-e)$

$x + \frac{1}{x} \geq 2 \Rightarrow \Rightarrow (1-e)\cdot\left(x + \frac{1}{x}\right) \leq 2(1-e)$

Το "=" ισχύει $\Leftrightarrow x = 1$

Άρα $g(x) \leq 2(1-e)$

$g(x) \leq \alpha \quad \forall x$

$\alpha \geq 2(1-e)$

$\frac{\alpha}{2} \geq 1 - e$

$\frac{\alpha}{2} + e \geq 1$

Scanned with CamScanner