$$\lim_{x \to +\infty} \left[x^2 \left(e^{\frac{1}{x}} - e^{\frac{1}{x+1}} \right) \right] = \lim_{x \to +\infty} \left[x^2 \cdot \frac{e^{\frac{1}{x}} - e^{\frac{1}{x+1}}}{\frac{1}{x} - \frac{1}{x+1}} \cdot \frac{1}{x^2+x} \right] = 1$$

$$\lim_{x \to +\infty} \left[\underbrace{\frac{x^2}{x^2+x}}_{A(x)} \cdot \underbrace{\frac{e^{\frac{1}{x}} - e^{\frac{1}{x+1}}}{\frac{1}{x} - \frac{1}{x+1}}}_{B(x)} \right]$$

$$\frac{1}{x} - \frac{1}{x+1} = \frac{x+1-x}{x \cdot (x+1)} = \frac{1}{x^2+x}$$

$$\lim_{x \to +\infty} A(x) = 1$$

Για την $f(x) = e^x$

Άρα Θ.Μ.Τ. $\Rightarrow \exists\, \xi \in \left(\frac{1}{x+1}, \frac{1}{x} \right)$: $B(x) = f'(\xi) = e^{\xi}$

$$\frac{1}{x+1} < \xi < \frac{1}{x} \Leftrightarrow e^{\frac{1}{x+1}} < e^{\xi} < e^{\frac{1}{x}} \Leftrightarrow e^{\frac{1}{x+1}} < B(x) < e^{\frac{1}{x}}$$

$$\lim_{x \to +\infty} e^{\frac{1}{x}} = \lim_{u \to 0^+} e^{u} = 1$$

$\lim_{x \to +\infty} e^{\frac{1}{x+1}} = 1$. Άρα ΚΠ $\Rightarrow$

$$\lim_{x \to +\infty} B(x) = 1$$

$$\lim_{x\to+\infty}\left(\sigma\upsilon\nu\sqrt{x+1}-\sigma\upsilon\nu\sqrt{x-1}\right)=\lim_{x\to+\infty}\left[\underbrace{\frac{\sigma\upsilon\nu\sqrt{x+1}-\sigma\upsilon\nu\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}}_{A(x)}\cdot\underbrace{\left(\sqrt{x+1}-\sqrt{x-1}\right)}_{B(x)}\right]=0$$

$$\lim_{x\to+\infty}B(x)=\lim_{x\to+\infty}\frac{\cancel{x}+1-\cancel{x}+1}{\sqrt{x+1}+\sqrt{x-1}}=0$$

Για την $f(x)=\sigma\upsilon\nu x$, από Θ.Μ.Τ $\Rightarrow\ \exists\ \xi\in\left(\sqrt{x-1},\sqrt{x+1}\right)$:

$$A(x)=f'(\xi)=-\eta\mu\xi \Rightarrow |A(x)|=|-\eta\mu\xi|\le 1 \Rightarrow$$

$$-1\le A(x)\le 1 \xrightarrow{B(x)>0}$$

$1>-1$

$x+1>x-1$

$\sqrt{x+1}>\sqrt{x-1}$

$B(x)>0$

$$\left.\begin{array}{l}-B(x)\le A(x)\cdot B(x)\le B(x)\\ \lim\limits_{x\to+\infty}B(x)=0\end{array}\right\}$$ Από κριτήριο Παρ. $\Rightarrow$

$$\lim_{x\to+\infty}\left[A(x)\cdot B(x)\right]=0$$

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Θέμα 3 / 2016 / ΕΜΕ.

$f(e^x+1) = x + e^x + 1 \quad (1) \quad \forall x \in \mathbb{R}$

α) Θέτω $u = e^x+1 \Rightarrow u-1 = e^x$, άρα $u-1>0 \Leftrightarrow u>1$

Τότε $x = \ln(u-1)$.

$(1) \Rightarrow f(u) = \ln(u-1) + u - \cancel{1} + \cancel{1} \Rightarrow f(u) = u + \ln(u-1),\ u>1$

β) $f'(x) = \dfrac{1}{x-1} + 1 = \dfrac{x}{x-1} > 0 \quad \forall x>1 \Rightarrow f \uparrow \Rightarrow 1-1$

f συνεχής και $\uparrow$ στο $A=(1,+\infty) \Rightarrow f(A) = \left(\lim\limits_{x\to1^+} f(x),\ \lim\limits_{x\to+\infty} f(x)\right) = \mathbb{R}$

$\lim\limits_{x\to1^+}\left[\ln(x-1)+x\right] \overset{u=x-1}{\underset{x=u+1}{=\!=}} \lim\limits_{u\to0^+}\left(\ln u + u+1\right) = -\infty.$ $\quad \lim\limits_{x\to+\infty} f(x) = +\infty \quad D_{f^{-1}} = f(A) = \mathbb{R}$

όταν $x\to1^+ \Rightarrow u\to0^+$

(γ) Λύνω την εξίσωση $f(x) = f^{-1}(x) \underset{\text{Λήμμα}}{\overset{f\uparrow}{\Longleftrightarrow}} f(x) = x \Leftrightarrow \ln(x-1)+x = x \Leftrightarrow \ln(x-1) = 0 \Leftrightarrow \boxed{x=2}$

Απόδ. Λήμματος | Έστω $A(x) = f(x)+x$. Έστω $x_1, x_2 \in (1,+\infty)$ με $x_1<x_2 \overset{f\uparrow}{\Longrightarrow} f(x_1)<f(x_2)$

$x_1 < x_2$

$A(x_1) < A(x_2)$

$f(x) = f^{-1}(x) \Leftrightarrow f(x)+x = f^{-1}(x)+x \Leftrightarrow A(x) = A(f^{-1}(x)) \Leftrightarrow$

$x = f^{-1}(x) \Leftrightarrow f(x) = x$