$f$ παρ/μη στο $\mathbb{R}$.

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \overset{\frac{0}{0}}{\underset{DLH}{=}} \lim_{x\to a} \frac{f'(x)}{1} = \lim_{x\to a} f'(x)$$

$\frac{0}{0}$, $\frac{\infty}{\infty}$, $0\cdot\infty$, $\infty-\infty$, $1^{\infty}$, $\infty^{0}$, $0^{0}$

ασκ. 2 (i) $$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = \lim_{x\to 0} e^{\frac{1}{x}\cdot\ln(1+x)} = \lim_{u\to 1} e^{u} = e$$

Θέτω $u = \frac{1}{x}\cdot\ln(1+x)$

όταν $x\to 0 \Rightarrow u \to 1$

$$\boxed{a^x = e^{x\ln a}}$$

$$\lim_{x\to 0}\frac{\ln(1+x)}{x} \overset{\frac{0}{0}}{\underset{DLH}{=}} \lim_{x\to 0} \frac{\frac{1}{1+x}\cdot\cancel{(1+x)}}{1} = 1$$

ασκ. 5 $f(0) = f'(0) = 0$, $f''(0) = 1$

$$\lim_{x\to 0}\frac{e^{f(x)}-1}{x^2} \overset{\frac{0}{0}}{\underset{DLH}{=}} \lim_{x\to 0}\frac{e^{f(x)}\cdot f'(x)}{2x} = \lim_{x\to 0}\frac{e^{f(x)}}{2}\cdot\lim_{x\to 0}\frac{f'(x)}{x} = \frac{1}{2}\cdot 1 = \frac{1}{2}$$

$$\lim_{x\to 0}\frac{f'(x)}{x} = \lim_{x\to 0}\frac{f'(x)-f'(0)}{x-0} = f''(0) = 1$$

άσκ. 3) $f'(0)=0$ μη σωστή.

i) $f(0)=1$

ii) $f$ παρ/μη στο $\mathbb{R}$.

$$f(x+y)=e^{2xy}\cdot f(x)\cdot f(y) \quad (1) \quad \forall x,y\in\mathbb{R}.$$

$x=y=0$ : $f(0)=e^{0}\cdot f^{2}(0) \Leftrightarrow f^{2}(0)=f(0)$ $\begin{cases} f(0)=0 \\ f(0)=1 \end{cases}$

Αν $f(0)=0$

$x$ $y=0$ $f(x)=e^{0}\cdot f(x)\cdot f(0)=0 \Rightarrow f(x)=0 \ \forall x\in\mathbb{R}$, άτοπο.

Άρα $f(0)=1$. $f'(0)=0 \Leftrightarrow \lim\limits_{x\to 0}\dfrac{f(x)-1}{x}=0$. $\lim\limits_{x\to 0}(f(x)-1)=0 \Leftrightarrow \lim\limits_{x\to 0}f(x)=1$

ii) Έστω τυχαίο $x_0\in\mathbb{R}$. $\lim\limits_{h\to 0}\dfrac{f(x_0+h)-f(x_0)}{h}=\lim\limits_{h\to 0}\dfrac{e^{2x_0\cdot h}\cdot f(x_0)\cdot f(h)-f(x_0)}{h}=$

$$=f(x_0)\lim_{h\to 0}\frac{e^{2x_0\cdot h}\cdot f(h)-1}{h}=f(x_0)\cdot\lim_{h\to 0}\Bigg[\underbrace{\frac{e^{2x_0\cdot h}\cdot f(h)-e^{2x_0\cdot h}}{h}}_{A(h)}+\underbrace{\frac{e^{2x_0\cdot h}-1}{h}}_{B(h)}\Bigg]=f(x_0)\left[0+2x_0\right]$$

$$2x_0 f(x_0)$$

$$\lim_{h\to 0}A(h)=\lim_{h\to 0}\left[e^{2x_0\cdot h}\,\frac{f(h)-1}{h}\right]=e^{0}\cdot f'(0)=0\ ,\quad \lim_{h\to 0}B(h)\ \overset{\frac{0}{0}}{\underset{DLH}{=}}\ \lim_{h\to 0}\frac{e^{2x_0\cdot h}\,2x_0}{1}=2x_0$$

iii) Από (ii) $\Rightarrow f'(x) = 2x \cdot f(x) \Leftrightarrow f'(x) - 2x \cdot f(x) = 0 \Leftrightarrow$

$f'(x) + (-x^2)' \cdot f(x) = 0 \Leftrightarrow f'(x) \cdot e^{-x^2} + e^{-x^2} \cdot (-x^2)' \cdot f(x) = 0 \Leftrightarrow$

$\left(f(x) \cdot e^{-x^2}\right)' = 0 \ldots$