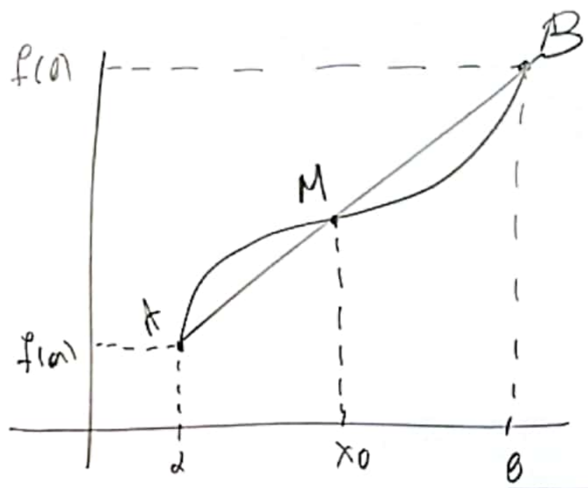
ασκ. 13



Από ΘΜΤ $\Rightarrow \exists\, \xi_1 \in (\alpha, x_0) : f'(\xi_1) = \dfrac{f(x_0)-f(\alpha)}{x_0-\alpha} = \lambda_{AM}$

$\Rightarrow \exists\, \xi_2 \in (x_0, \beta) : f'(\xi_2) = \dfrac{f(\beta)-f(x_0)}{\beta-x_0} = \lambda_{MB}$

A, M, B συνευθειακά $\Rightarrow \lambda_{AM} = \lambda_{MB} \Rightarrow f'(\xi_1) = f'(\xi_2)$

Από Θ. Rolle $\Rightarrow \exists\, \xi \in (\xi_1, \xi_2) : f''(\xi) = 0$

ασκ. 8 $f' \uparrow$ στο Δ. $\forall \alpha, \beta \in \Delta$: $f(\alpha)+f(\beta) \geq 2 \cdot f\left(\dfrac{\alpha+\beta}{2}\right) \overset{(1)}{\Leftrightarrow} \dfrac{f(\alpha)+f(\beta)}{2} \geq f\left(\dfrac{\alpha+\beta}{2}\right)$

$(1) \Leftrightarrow f(\beta) - f\left(\dfrac{\alpha+\beta}{2}\right) \geq f\left(\dfrac{\alpha+\beta}{2}\right) - f(\alpha).$ (2)

1η περ. Αν $\alpha = \beta$ τότε $(1) \Leftrightarrow$ $2f(\alpha) \geq 2f\left(\dfrac{2\alpha}{2}\right)$ ισχύει σαν ισότητα

Τώρα: Αν $\alpha < \beta$:

Από ΘΜΤ $\Rightarrow \xi_1 \in \left(\alpha, \dfrac{\alpha+\beta}{2}\right) : f'(\xi_1) = \dfrac{f\left(\frac{\alpha+\beta}{2}\right)-f(\alpha)}{\frac{\alpha+\beta}{2}-\alpha} = \dfrac{f\left(\frac{\alpha+\beta}{2}\right)-f(\alpha)}{\frac{\beta-\alpha}{2}}$

Από ΘΜΤ $\Rightarrow \xi_2 \in \left(\dfrac{\alpha+\beta}{2}, \beta\right) : f'(\xi_2) = \dfrac{f(\beta)-f\left(\frac{\alpha+\beta}{2}\right)}{\beta-\frac{\alpha+\beta}{2}} = \dfrac{f(\beta)-f\left(\frac{\alpha+\beta}{2}\right)}{\frac{\beta-\alpha}{2}}$

$\alpha < \xi_1 < \dfrac{\alpha+\beta}{2} < \xi_2 < \beta \Rightarrow \xi_1 < \xi_2 \xrightarrow{f' \uparrow} f'(\xi_1) < f'(\xi_2) \Leftrightarrow \dfrac{f\left(\frac{\alpha+\beta}{2}\right)-f(\alpha)}{\cancel{\frac{\beta-\alpha}{2}}} < \dfrac{f(\beta)-f\left(\frac{\alpha+\beta}{2}\right)}{\cancel{\frac{\beta-\alpha}{2}}}$ όπου

Δείξαμε την $(2) \Leftrightarrow (1)$

ασκ 6 / Σ.132 / Σχολικό

$f'(x) \le 1 \quad \forall x \in (-1,1)$

$f(-1) = -1, \quad f(1) = 1$

Ν.δ.ο. $f(0) = 0$.

Ν.δ.ο $f(x) = x \quad \forall x \in [-1,1]$

Από Θ.Μ.Τ. $\Rightarrow \exists \xi_1 \in (-1,0): \dfrac{f(0)-f(-1)}{1} = f'(\xi_1) \Rightarrow$

$f(0)+1 = f'(\xi_1) \overset{(\text{Y})}{\le} 1 \Rightarrow f(0)+1 \le 1 \Leftrightarrow \boxed{f(0) \le 0 \quad (1)}$

Από Θ.Μ.Τ. $\Rightarrow \exists \xi_2 \in (0,1): \dfrac{f(1)-f(0)}{1-0} = f'(\xi_2) \Leftrightarrow$

$1 - f(0) = f'(\xi_2) \overset{(\text{Y})}{\le} 1 \Rightarrow 1 - f(0) \le 1 \Leftrightarrow -f(0) \le 0 \Leftrightarrow$

$\boxed{f(0) \ge 0 \quad (2)}$

Από Θ.Μ.Τ. $\Rightarrow \exists \xi_1 \in (-1,x): f'(\xi_1) = \dfrac{f(x)-f(-1)}{x+1} \le 1 \Rightarrow \dfrac{f(x)+1}{x+1} \le 1 \overset{x+1>0}{\Leftrightarrow} f(x)+1 \le x+1$

" " $\exists \xi_2 \in (x,1): f'(\xi_2) = \dfrac{f(1)-f(x)}{1-x} \le 1 \Rightarrow \ldots$

$\left.\begin{array}{l} \Rightarrow f(x) \le x \\ f(x) \ge x \end{array}\right\} \Rightarrow f(x) = x$

2ος | Με απαγωγή σε άτοπο: Έστω $\exists x_1 \in (-1,1): f(x_1) \ne x_1 \Rightarrow f(x_1) < x_1$ ή $f(x_1) > x_1$

Από Θ.Μ.Τ. $\Rightarrow \exists \xi_1 \in (-1,x_1): \dfrac{f(x_1)-f(-1)}{x_1+1} = f'(\xi_1) \le 1 \Rightarrow \dfrac{f(x_1)+1}{x_1+1} \le 1 \Rightarrow$

$f(x_1)+1 \le x_1+1 \Rightarrow f(x_1) \le x_1$

Από Θ.Μ.Τ $\Rightarrow \exists \xi_2 \in (x_1,1): \dfrac{f(1)-f(x_1)}{1-x_1} = f'(\xi_2) \le 1 \Rightarrow \dfrac{1-f(x_1)}{1-x_1} \le 1 \Leftrightarrow 1-f(x_1) \le 1-x_1 \overset{-1}{\Leftrightarrow}$

$-f(x_1) \le -x_1 \Leftrightarrow f(x_1) \ge x_1$

$\underline{\text{Άσκ 9}}$ $a, b \in \left(0, \frac{\pi}{2}\right)$ $a < b \Rightarrow \dfrac{b-a}{\eta\mu^2 b} < \sigma\varphi a - \sigma\varphi b < \dfrac{b-a}{\eta\mu^2 a}$ (1) $\Leftrightarrow$

$$\frac{1}{\eta\mu^2 b} < \frac{\sigma\varphi a - \sigma\varphi b}{b-a} < \frac{1}{\eta\mu^2 a} \Leftrightarrow -\frac{1}{\eta\mu^2 b} > \frac{\sigma\varphi b - \sigma\varphi a}{b-a} > \frac{-1}{\eta\mu^2 a} \quad (2)$$

$f(x) = \sigma\varphi x$. Από ΘΜΤ $\Rightarrow \exists\, \xi \in (a, b)$: $\dfrac{\sigma\varphi b - \sigma\varphi a}{b-a} = -\dfrac{1}{\eta\mu^2 \xi}$ (3)

$$0 < a < \xi < b < \frac{\pi}{2} \xrightarrow{\eta\mu x \uparrow} 0 < \eta\mu a < \eta\mu \xi < \eta\mu b \Rightarrow \eta\mu^2 a < \eta\mu^2 \xi < \eta\mu^2 b \Leftrightarrow$$

$$\frac{1}{\eta\mu^2 a} > \frac{1}{\eta\mu^2 \xi} > \frac{1}{\eta\mu^2 b} \Leftrightarrow -\frac{1}{\eta\mu^2 a} < -\frac{1}{\eta\mu^2 \xi} < -\frac{1}{\eta\mu^2 b} \overset{(3)}{\Leftrightarrow}$$

$$-\frac{1}{\eta\mu^2 a} < \frac{\sigma\varphi b - \sigma\varphi a}{b-a} < -\frac{1}{\eta\mu^2 b}.$$ Ισχύει η (2) $\Leftrightarrow$ (1)

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ασκ 17 / (ii) $1-\frac{1}{x} < \ln x < x-1$ (1), $\forall x>1$

(1) $\Leftrightarrow \frac{x-1}{x} < \ln x < x-1 \Leftrightarrow \frac{1}{x} < \frac{\ln x - \ln 1}{x-1} < 1$ (2)

Έστω $f(x)=\ln x$. Από Θ.Μ.Τ. $\Rightarrow \exists\, \xi \in (1,x): \frac{\ln x-\ln 1}{x-1} = \frac{1}{\xi}$ (3)

Αφού $1<\xi<x \Rightarrow 1 > \frac{1}{\xi} > \frac{1}{x} \Rightarrow$

$1 > \frac{\ln x - \ln 1}{x-1} > \frac{1}{x}$ ο.ε.δ.

Σχόλιο. $\ln x \le x-1$ $\forall x>0$ και το "=" ισχύει μόνο για $x=1$.

Για $x>1 \Rightarrow \frac{1}{x}<1$. $\ln\frac{1}{x} \le \frac{1}{x}-1 \Leftrightarrow -\ln x \le \frac{1}{x}-1 \Leftrightarrow \ln x \geq 1-\frac{1}{x}$

(i) $1 < \frac{e^x-1}{x} < e^x$, $x>0$. Έστω $f(x)=e^x$. ΘΜΤ στο $(0,x)$

2ος τρόπος. $1<\frac{e^x-1}{x}<e^x$ (1) $\Leftrightarrow x < e^x-1 < x\cdot e^x$

θ.δ.ο. $e^x-1 < x\cdot e^x$ (4) $\Leftrightarrow$
$1-e^{-x} < x$ (5)

$x=e^x$: $\ln e^x \le e^x-1 \Leftrightarrow x \le e^x-1$
το "=" ισχύει $\Leftrightarrow e^x=1 \Leftrightarrow x=0$

Έχω δείξει $x<e^x-1 \Rightarrow -x < e^{-x}-1 \Rightarrow$
$x > -e^{-x}+1$ ο.ε.δ.

ασκ 17

$$1 < \frac{e^x - 1}{x} < e^x$$

$f(x) = e^x$. Από Θ.Μ.Τ. $\Rightarrow$ $\exists\, \xi \in (0, x)$ ώστε $\dfrac{e^x - e^0}{x - 0} = e^{\xi} \Leftrightarrow \dfrac{e^x - 1}{x} = e^{\xi}$

$$0 < \xi < x \Rightarrow e^0 < e^{\xi} < e^x$$

ασκ. 15

$$\frac{2x}{1+x} < \ln\left(\frac{1+x}{1-x}\right) < \frac{2x}{1-x} \quad (1), \quad \forall x \in (0,1)$$

$$(1) \Leftrightarrow \frac{2x}{1+x} < \ln(1+x) - \ln(1-x) < \frac{2x}{1-x} \Leftrightarrow \frac{1}{1+x} < \frac{\ln(1+x) - \ln(1-x)}{2x} < \frac{1}{1-x}$$

$$(1+x) - (1-x) = 1 + x - 1 + x = 2x$$

Έστω $f(x) = \ln x$

Από Θ.Μ.Τ. $\Rightarrow$ $\exists\, \xi \in (1-x, 1+x)$: $\dfrac{\ln(1+x) - \ln(1-x)}{2x} = \dfrac{1}{\xi}$

$$0 < 1-x < \xi < 1+x \Leftrightarrow \frac{1}{1-x} > \frac{1}{\xi} > \frac{1}{1+x}$$

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