

$f$  strictly increasing on  $[a, b]$ ,  $x_0 \in (a, b)$

$f \nearrow$  on  $[a, x_0)$ ,

$$a \leq x_1 < x_2 < x_0$$

$f \nearrow$  on  $[x_0, b]$

$$x_0 \leq x_1 < x_2 < b$$

N.B.O.  $f \nearrow$  on  $[a, b]$

$$x_1 < x_0 < x_2$$

$$x_1 < x_0 \implies f(x_1) < f(x_0)$$

$$\lim_{x \rightarrow x_0^-} f(x) = f(x_0)$$

$$f([a, x_0]) = [f(a), f(x_0)] \quad \lim_{x \rightarrow x_0^-} f(x)$$

$$f(x_1) \in [f(a), f(x_0)]$$

$$f(x_1) < f(x_0)$$

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f(0) = 1

f'(x) - x · f(x) = x + 1 (1) ∀ x ∈ ℝ

i) f'(-2) + 2f(-2) = -1 ⇔ f'(-2) + 2f(-2) + 1 = 0 ⇔ (f(-2) + 1)² = 0 ⇔

f(-2) = -1

ii) f has root in [-2, 0] (⇒ Ans ∅? ⇒ ∃ x₀ ∈ (-2, 0): f(x₀) = 0.

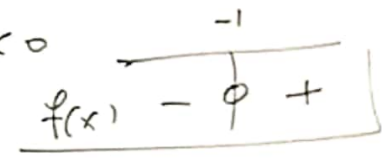
f(0) ≠ f(-2)

0 ∈ (f(-2), f(0))

(1) x = x₀, f'(x₀) - x₀ · f(x₀) = x₀ + 1 ⇔ x₀ + 1 = 0 ⇔ x₀ = -1

(iii) f has root in (-∞, -1), (-1, +∞) can't be proven by other.

Ans using Bolzano ⇒ diamphi problem. f(0) = 1 > 0, f(-2) = -1 < 0



**235**  $f(0) = 1$ ,  $f^2(x) = x^2 f(x) + x^2 + a \quad (1) \quad \forall x \in \mathbb{R}$

(i)  $f^2(0) = a \Rightarrow a = 1$ .

(ii) Έστω  $x_0 \neq 0 \Rightarrow f(x_0) = 0 \xrightarrow{(1)} f^2(x_0) = x_0^2 f(x_0) + x_0^2 + 1 \Leftrightarrow x_0^2 = -1$   
 άτοπο.

(iii)  $f(x) \neq 0 \quad \forall x \in \mathbb{R}$  } Ανι ξηθάιο Bolzano  $\Rightarrow$  διαμπερι πρόβηηο ένο  $\mathbb{R}$ .  
 $f$  έμης ένο  $\mathbb{R}$  } Απει  $f(0) = 1 > 0 \Rightarrow f(x) > 0 \quad \forall x \in \mathbb{R}$

(iv)  $f^2(x) = x^2 f(x) + x^2 + 1 \Leftrightarrow$

$f^2(x) - x^2 f(x) = x^2 + 1 \Leftrightarrow$

$f^2(x) - 2 \cdot \frac{1}{2} x^2 f(x) + \frac{1}{4} x^4 = \frac{1}{4} x^4 + x^2 + 1 \Leftrightarrow$

$\left( f(x) - \frac{1}{2} x^2 \right)^2 = \left( \frac{1}{2} x^2 + 1 \right)^2 \quad (2)$   
 $\parallel$   
 $A(x)$

$\forall A(x) \neq 0 \quad \forall x \in \mathbb{R}$  }  $\rightarrow$  άνη έη. Bolz  
 $A(x)$  έμης ένο  $\mathbb{R}$  }  $\rightarrow$  διαμπερι πρόβηηο

$A(0) = f(0) - \frac{1}{2} \cdot 0^2 = 1 > 0$

$\Rightarrow A(x) > 0 \quad \forall x \in \mathbb{R}$ .

$\left| f(x) - \frac{1}{2} x^2 \right| = \frac{1}{2} x^2 + 1 \quad (2) \Rightarrow$

$f(x) - \frac{1}{2} x^2 = \frac{1}{2} x^2 + 1 \Leftrightarrow$

$f(x) = x^2 + 1, \quad x \in \mathbb{R}.$