Ένα σημαντικό λήμμα

Αν $\lim\limits_{x\to x_0}\dfrac{f(x)}{g(x)} = \ell \in \mathbb{R}$ και $\lim\limits_{x\to x_0} g(x) = 0$, τότε $\lim\limits_{x\to x_0} f(x) = 0$.

Απόδειξη: Θέτω $A(x) = \dfrac{f(x)}{g(x)}$, ξέρω $\lim\limits_{x\to x_0} A(x) = \ell \in \mathbb{R}$

$\Leftrightarrow f(x) = A(x)\cdot g(x)$. $\quad \lim\limits_{x\to x_0}\left[A(x)\cdot g(x)\right] = \lim\limits_{x\to x_0} A(x)\cdot \lim\limits_{x\to x_0} g(x) = \ell\cdot 0 = 0$

Άρα $\lim\limits_{x\to x_0} f(x) = 0$.

Ασκ.4 $\quad \lim\limits_{x\to 0}\dfrac{|ημx\cdot P(x)+1| - |ημx\cdot P(x)-1|}{x} = 2$, $P(x)$ πολυώνυμο, σταθερός όρος $=$;

$\lim\limits_{x\to 0}\left(ημx\cdot P(x)+1\right) = 0\cdot P(0)+1 = 1 > 0 \Rightarrow ημx\cdot P(x)+1 > 0$ κοντά στο 0.

$\lim\limits_{x\to 0}\left(ημx\, P(x)-1\right) = -1 < 0 \Rightarrow ημx\cdot P(x)-1 < 0$ κοντά στο 0.

Άρα $\lim\limits_{x\to 0}\dfrac{ημx\cdot P(x)+1-\left(-ημx\cdot P(x)+1\right)}{x} = 2 \Leftrightarrow \lim\limits_{x\to 0}\dfrac{2ημx\cdot P(x)}{x} = 2 \Leftrightarrow 2\cdot 1\cdot P(0) = 2 \Leftrightarrow$

$P(0) = 1 \Rightarrow$ ο σταθερός όρος $= 1$

άσκ. 8 / Αν $\lim_{x\to 0} \frac{f(x)}{x} = 2$, α) $\lim_{x\to 0} \frac{f(x)+\varepsilon\varphi x}{2f(x)+1-\sigma\upsilon\nu x} = \lim_{x\to 0} \frac{\frac{f(x)}{x}+\frac{\varepsilon\varphi x}{x}}{\frac{2f(x)}{x}+\frac{1-\sigma\upsilon\nu x}{x}} = \frac{2+1}{2 2+0} = \frac{3}{4}$

$$\lim_{x\to 0} \frac{\sqrt{3f(x)+1}-1}{5f(x)+x+\sigma\upsilon\nu x} = \frac{\sqrt{3\cdot 0+1}-1}{5\cdot 0+0+1} = \frac{0}{1} = 0.$$

Αφού $\left.\begin{array}{l}\lim_{x\to 0} \frac{f(x)}{x} = 2 \\ \lim_{x\to 0} x = 0\end{array}\right\} \xrightarrow[\text{ιδιότ.}]{\text{πράξ.}} \lim_{x\to 0} f(x) = 0$

άσκ. 6 α) $\lim_{x\to 0^+} \frac{x\cdot\eta\mu\frac{1}{x}}{x+\sqrt{|x|}} = \lim_{x\to 0^+} \frac{x\cdot\eta\mu\frac{1}{x}}{x+\sqrt{x}} = \lim_{x\to 0^+} \frac{x\cdot\eta\mu\frac{1}{x}}{\sqrt{x}\cdot(\sqrt{x}+1)} = \lim_{x\to 0^+} \frac{\sqrt{x}\cdot\eta\mu\frac{1}{x}}{\sqrt{x}+1} = 0$

Θεωρώ $\left|\sqrt{x}\cdot\eta\mu\frac{1}{x}\right| = \sqrt{x}\cdot\left|\eta\mu\frac{1}{x}\right| \le \sqrt{x}\cdot 1 \Rightarrow -\sqrt{x} \le \sqrt{x}\cdot\eta\mu\frac{1}{x} \le \sqrt{x}$

$$\lim_{x\to 0^-} \frac{x\cdot\eta\mu\frac{1}{x}}{x+\sqrt{-x}} = \lim_{x\to 0^-} \frac{-\sqrt{-x}^{\,2}\cdot\eta\mu\frac{1}{x}}{-\sqrt{-x}^{\,2}+\sqrt{-x}} = \lim_{x\to 0^-} \frac{-\sqrt{-x}^{\,\cancel{2}}\cdot\eta\mu\frac{1}{x}}{\cancel{\sqrt{-x}}\cdot(-\sqrt{-x}+1)} = \frac{0}{1} = 0$$

Άσκ. 3 | (θ) $\lim_{x\to 0} \dfrac{x^k \cdot ημ\frac{1}{x}}{ημx} = ?$ $\quad k \geq 2$

$$= \lim_{x\to 0}\left(\frac{x}{ημx}\cdot x^{k-1}\cdot ημ\frac{1}{x}\right) = 1 \cdot 0 = 0$$

$$\lim_{x\to 0} \frac{x}{ημx} = 1. \qquad \left|x^{k-1}\cdot ημ\frac{1}{x}\right| = |x^{k-1}|\cdot\left|ημ\frac{1}{x}\right| \leq |x^{k-1}|$$

$$\underbrace{-|x^{k-1}|}_{\to 0} \leq x^{k-1} ημ\frac{1}{x} \leq \underbrace{|x^{k-1}|}_{\to 0}$$

$$0$$

ι) $$\lim_{x\to 1} \frac{ημ(x-1)}{x^2+4x-5} = \lim_{x\to 1}\left(\frac{ημ(x-1)}{x-1}\cdot\frac{x-1}{x^2+4x-5}\right)$$